Titolo del corso: **Optimal Transport and Applications**

Docente: Cristian Gutierrez

Membro del collegio proponente: Giovanni Cupini

Ore frontali di lezione: 12

Periodo di lezione: settembre-ottobre 2026

Settore/i disciplinare del corso: MAT/05

Tipologia di corso: Base

Modalità di verifica dell'apprendimento:

Abstract del corso:

Programma del corso:

1. **Optimal transport and the Wasserstein metric.** Formulation and main results concerning the Monge and Kantorovich problems, along with the connections between them. The Wasserstein distance and its dynamic formulation by Benamou–Brenier: continuity equation, regularity properties, and convexity in spaces of measures. The multimarginal Monge problem and some of its applications.

2. **Metasurfaces and time-varying media**. Generalized Snell's law derived from Maxwell's equations in the distributional sense. Temporal interfaces: reflection and transmission factors, energy conservation and variation. Associated problems and their connections with multimarginal optimal transport.

OPTIMAL TRANSPORT AND APPLICATIONS
UNIVERSITY OF BOLOGNA
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Optimal mass transportation pertains to the optimal allocation of resources. For instance, allocating individuals to jobs, transporting goods from warehouses to retail outlets, and transforming one image into another. In each of these scenarios, a function is provided that represents the cost of mapping or transporting a unit of one item into another item. The objective is to determine a method for simultaneously allocating all resources to minimize the total cost. This question originated with the work of Monge in the 18th century, who sought to solve a problem proposed by engineers (linear cost). It remained dormant until 1940 when Kantorovich, motivated by economic problems, discovered a probabilistic formulation.

In addition to being a crucial component in the development of these concepts, linear programming (LP) was conceived during World War II to address planning challenges in wartime operations. In the post-war era, numerous industries recognized the significant utility of these advancements in optimizing their business processes, and these tools continue to be employed to this day. Notable figures and pioneers in LP include George Dantzig, who pioneered the simplex method, John von Neumann, who established the duality theory, and Thomas Koopmans, who applied LP to economic contexts. In 1975, Kantorovich and Koopmans were honored with the Nobel Prize in Economics for their contributions to the optimal allocation of resources.

The subject has experienced significant growth in the past three decades, establishing mathematical connections with convex analysis, optimization, probability, and partial differential equations (PDEs). Notably, it has been discovered to possess applications in various fields, including optics, image processing, and machine learning.

The objective of this course is to elucidate certain theoretical concepts and illustrate their practical applications. Students will be assigned study projects.

Topics include (time permitting):

(1) Monge and Kantorovich problems; existence of optimal maps/plans; use of convex analysis; dual problems.
(2) Sinkhorn's Theorem and Application to the Distribution Problem
(3) Introduction to Monge-Ampère equations.
(4) Brenier's polar factorization theorem.

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(5) Monge-Kantorovich distance (or Wasserstein distance).
(6) Benamou-Brenier dynamic formulation of optimal transport.
(7) Multi-marginal Monge problem.
(8) Applications to geometric optics
(9) Generalized Snell's law and metasurfaces.

This course will be of particular interest to students pursuing studies in analysis, probability, applied mathematics, and economics. Prerequisites include a solid understanding of real analysis, basic partial differential equations (PDEs), abstract measure theory, and basic concepts in functional analysis.

References

[Dan63] George B. Dantzig, *Linear Programming and Extensions*, Report R-366-PR, The Rand Corporation, Santa Monica, CA, August 1963.

[GH09] C. E. Gutiérrez and Qingbo Huang, *The refractor problem in reshaping light beams*, Arch. Rational Mech. Anal. **193** (2009), no. 2, 423–443.

[GH14] ______, *The near field refractor*, Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire **31** (2014), no. 4, 655–684.

[Gut16] C. E. Gutiérrez, *The Monge–Ampère equation*, 2nd ed., Progress in Nonlinear Differential Equations and Their Applications, vol. 89, Birkhäuser, Boston, MA, 2016.

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[Lun64] R. K. Luneburg, *Mathematical theory of optics*, University of California Press, Berkeley and L.A., CA, 1964.

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[Vil08] C. Villani, *Optimal transport, old and new*, https://cedricvillani.org/for-mathematicians/surveys-books/, 2008.

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